On the integrality of the elementary symmetric functions of $1, 1/3, ..., 1/(2n-1)$ *

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Abstract

Erdős and Niven proved that for any positive integers m and d, there are only finitely many positive integers n for which one or more of the elementary symmetric functions of $1/m, 1/(m+d), ..., 1/(m+nd)$ are integers. Recently, Chen and Tang proved that if $n \geq 4$, then none of the elementary symmetric functions of $1, 1/2, ..., 1/n$ is an integer. In this paper, we show that if $n \geq 2$, then none of the elementary symmetric functions of $1, 1/3, ..., 1/(2n-1)$ is an integer.

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1. Introduction

A well-known result in number theory says that for any positive integers m, d, if $n > 1$, then the harmonic sum $\sum_{i=1}^{n} \frac{1}{m+id}$ is not an integer. In 1946, Erdős and Niven [1] proved that there are only finitely many integers n for which one or more of the elementary symmetric functions of

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$1, 1/2, ..., 1/n$ are integers, and they mentioned that by a similar argument one could acquire the same result for the elementary symmetric function of $1/m, 1/(m+d), ..., 1/(m+nd)$ for any given positive integers m and d. Recently, Chen and Tang [3] proved that if $n \geq 4$, then none of the elementary symmetric functions of $1, 1/2, ..., 1/n$ is not an integer. It is an interesting question to determine all finite arithmetic progressions $\{m+di\}_{i=0}^{n}$ such that one or more elementary symmetric functions of $1/m, 1/(m+d), ..., 1/(m+nd)$ are integers.

In this paper, we consider the finite arithmetic progression $\{1+2i\}_{i=0}^{n-1}$. Throughout, we let $S_k(n)$ denote the k-th elementary symmetric functions of $1, 1/3, ..., 1/(2n-1)$. That is,

$$S_k(n) := \sum_{0 \leq i_1 < ... < i_k \leq n-1} \prod_{j=1}^{k} \frac{1}{(1+2i_j)}.$$

We will show that for all integers $n > 1$ and $1 \leq k \leq n$, $S_k(n)$ is not an integer. See Theorem 3.1 below. The paper is organized as follows. In Section 2, we show several lemmas which are needed for the proof of the main result. In the last section, we give the main result.

2. Several lemmas

In the present section, we show some preliminary lemmas which are needed for the proof of our main result. As usual, let $\pi(x)$ denote the number of primes no more than x. We begin with a known result.

Lemma 2.1. [2] *One has*

$$\pi(x) < \frac{x}{\log x - 1 - (\log x)^{-1/2}} \text{ for all } x \geq 6$$

and

$$\pi(x) > \frac{x}{\log x - 1 + (\log x)^{-1/2}} \text{ for all } x \geq 59.$$

Lemma 2.2. *For any integer $k \geq 1$, we have*

$$S_k(k+1) = \frac{(k+1)^2}{\prod_{i=0}^{k}(1+2i)}, S_k(k+2) = \frac{(k+1)(k+2)(3k^2+11k+9)}{6\prod_{i=0}^{k+1}(1+2i)}$$

and

$$S_k(k+3) = \frac{(k+1)(k+2)(k+3)^2(k^2+5k+5)}{6\prod_{i=0}^{k+2}(1+2i)}.$$

Proof. Since

$$\sum_{i=0}^{k}(1+2i) = (k+1)^2, \sum_{0\le i<j\le k+1}(1+2i)(1+2j) = \frac{1}{6}(k+1)(k+2)(3k^2+11k+9)$$

and

$$\sum_{0\le i<j<l\le k+2}(1+2i)(1+2j)(1+2l) = \frac{1}{6}(k+1)(k+2)(k+3)^2(k^2+5k+5).$$

the desired formulae follow immediately. So Lemma 2.2 is proved. □

Lemma 2.3. *Let k and n be positive integers such that*

$$e\left(\frac{1}{2}\log(2n-1)+1\right) \le k \le n.$$

Then $S_k(n)$ is not an integer.

Proof. First by the multi-nomial expansion theorem, we get

$$S_k(n) \le \frac{1}{k!}\left(\sum_{i=0}^{n-1}\frac{1}{1+2i}\right)^k.$$

On the one hand, one has

$$\sum_{i=0}^{n-1}\frac{1}{1+2i} < 1 + \int_0^{n-1}\frac{1}{1+2x}dx = \frac{1}{2}\log(2n-1)+1.$$

On the other hand, we have

$$\log k! = \sum_{i=2}^{k}\log i > \int_1^k \log x\mathrm{d}x > k\log k - k > k\log(\frac{1}{2}\log(2n-1)+1).$$

So from the above inequalities, we deduce that

$$\left(\sum_{i=0}^{n-1}\frac{1}{1+2i}\right)^k < k!.$$

In other words, $S_k(n) < 1$ if $n \geq k \geq e(\frac{1}{2}\log(2n-1)+1)$. This ends the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let k and n be positive integers such that $1 < k \leq n$. Suppose that there exists an odd prime $p > 2k+6$ satisfying that*

$$\frac{n}{k+3} < p \leq \frac{n}{k}$$

and

$$p \nmid (3k^2+11k+9)(k^2+5k+5).$$

Then $S_k(n)$ is not an integer.

Proof. First of all, we can easily check that the following identity holds:

$$S_k(n) = \sum_{0 \leq i_1 < \cdots < i_k \leq [\frac{n}{p}]+t} \prod_{j=1}^{k} \frac{1}{p(1+2i_j)} + \sum_{\substack{0 \leq i_1 < \cdots < i_k \leq n-1 \\ \exists j \text{ s.t. } p \nmid (1+2i_j)}} \prod_{j=1}^{k} \frac{1}{(1+2i_j)}, \tag{2.1}$$

where $t = -1$ if $p(1+2[\frac{n}{p}]) > 2n-1$, and $t = 0$ otherwise.

Since $p > 2k+6$ and $p > \frac{n}{k+3}$, we have $p > \sqrt{2n-1}$. It infers that $v_p(1+2i) \leq 1$ for $0 \leq i \leq n-1$, where v_p denotes the p-adic valuation on $\mathbb{Q}$. We then derive from (2.1) that

$$S_k(n) = \frac{1}{p^k} S_k\left(\left[\frac{n}{p}\right]+t+1\right) + \frac{a}{p^{k-1}b} \tag{2.2}$$

for some positive integers a and b with $p \nmid b$. Note that $k \leq [\frac{n}{p}]+t+1 \leq k+3$. But $p > 2k+6$ and $p \nmid (3k^2+11k+9)(k^2+5k+5)$. Then by Lemma 2.2 we obtain that $v_p(S_k([\frac{n}{p}]+t+1)) = 0$.

Now using (2.2), we can get that $v_p(S_k(n)) = -k < 0$. Therefore $S_k(n)$ is not an integer as desired. The proof of Lemma 2.4 is complete. $\square$

3. The main result

In this section, we give the main result of this paper.

Theorem 3.1. *For any integers $n > 1$ and k with $1 \leq k \leq n$, $S_k(n)$ is not an integer.*

Proof. When $k = 1$, it is known that for any integer $n > 1$, $\sum_{i=0}^{n-1} \frac{1}{1+2i}$

is not an integer (see, for example, [1]). So Theorem 3.1 is true when $k = 1$. In what follows we let $k \geq 2$.

By Lemma 2.3, we know that $S_k(n)$ is not an integer if $e(\frac{1}{2}\log(2n-1)+1) \leq k \leq n$. In the following we assume that $2 \leq k < e(\frac{1}{2}\log(2n-1)+1)$.

First we let $n \geq 23000$. Claim that there is a prime number $p > 2k+6$ such that $\frac{n}{k+3} < p \leq \frac{n}{k}$ and $p \nmid (3k^2+11k+9)(k^2+5k+5)$. It then follows immediately from the claim and Lemma 2.4 that $S_k(n)$ is not an integer for all $2 \leq k < e(\frac{1}{2}\log(2n-1)+1)$ if $n \geq 23000$. It remains to show the claim which will be done in the following.

First we prove $\pi(\frac{n}{k}) > \pi(\frac{n}{k+3})$. By Lemma 2.1, it suffices to show that

$$\frac{n/k}{\log(n/k) - 1 + (\log(n/k))^{-1/2}} > \frac{n/(k+3)}{\log(n/(k+3)) - 1 - (\log(n/(k+3)))^{-1/2}}.$$

This is equivalent to

$$k\log\left(1+\frac{3}{k}\right) + 3 + k\left(\log\frac{n}{k}\right)^{-1/2} + (k+3)\left(\log\frac{n}{k+3}\right)^{-1/2} < 3\log\frac{n}{k+3}. \tag{3.1}$$

Since $k\log(1+\frac{3}{k}) < 3$ and $\log\frac{n}{k} > \log\frac{n}{k+3}$, in order to show that (3.1) holds, it is enough to prove that the following inequality is true:

$$6 + (2k+3)\left(\log\frac{n}{k+3}\right)^{-1/2} < 3\log\frac{n}{k+3}. \tag{3.2}$$

Define a real function $f(x)$ by

$$f(x) := x^{0.3} - \frac{e}{2}\log(2x-1) - e - 3.$$

Then one can easily check that $f(23000) > 0$ and

$$xf'(x) = 0.3x^{0.3} - \frac{ex}{2x-1} > 0$$

for all $x \geq 23000$. We can derive that $f(x) > 0$ for all $x \geq 23000$. But $k < e(\frac{1}{2}\log(2n-1)+1)$. So for $n \geq 23000$, we have

$$\frac{n}{k+3} > \frac{n}{\frac{e}{2}\log(2x-1)+e+3} > n^{0.7}.$$

Thus to prove (3.2), it is sufficient to show the following inequality

$$6 + (e\log(2n-1) + 2e + 3)(\log n^{0.7})^{-1/2} \leq 3\log n^{0.7}, n \geq 23000,$$

which is equivalent to

$$6\times 0.7^{1/2}(\log n)^{1/2}+e\log(2n-1)+2e+3\le 3\times 0.7^{3/2}(\log n)^{3/2}, n\ge 23000. \tag{3.3}$$

Let $t=(\log n)^{1/2}$. Then one find that for $t\ge 3.10$, we have

$$6\times 0.7^{1/2}t+et^2+e\log 2+2e+3\le 3\times 0.7^{3/2}t^3,$$

from which (3.3) follows immediately. Hence (3.2) is proved and so we have $\pi(\frac{n}{k})>\pi(\frac{n}{k+3})$ for $k<e(\frac{1}{2}\log(2n-1)+1)$ if $n\ge 23000$.

Consequently, we prove that $\frac{n}{k+3}>3k^2+11k+9$ for $k<e(\frac{1}{2}\log(2n-1)+1)$ if $n\ge 23000$. Evidently we have $n>\frac{1}{2}e^{(2k/e)-2}$ since $k<e(\frac{1}{2}\log(2n-1)+1)$. It is easy to show that

$$\frac{1}{2}e^{(2x/e)-2}>g(x):=(x+3)(3x^2+11x+9)$$

for all $x\ge 17.3$. Let $h(x):=e(\frac{1}{2}\log(2x-1)+1)$. Then $h(n)\ge 17.3$ if $n\ge 23000$ and $h(n)>k$. It follows that

$$n=\frac{1}{2}e^{\frac{2h(n)}{e}-2}+\frac{1}{2}>\frac{1}{2}e^{\frac{2h(n)}{e}-2}>g(h(n))>g(k).$$

Namely, we have $\frac{n}{k+3}>3k^2+11k+9$ for $k<e(\frac{1}{2}\log(2n-1)+1)$ if $n\ge 23000$.

Since $\pi(\frac{n}{k})>\pi(\frac{n}{k+3})$, there is a prime number p satisfying $\frac{n}{k+3}<p\le\frac{n}{k}$. But $\frac{n}{k+3}>3k^2+11k+9$. Thus $p>2k+6$ and $p\nmid(3k^2+11k+9)(k^2+5k+5)$. Hence the claim is proved.

Now we treat the remaining case: $n<23000$. Since $k<e(\frac{1}{2}\log(2n-1)+1)$, we have $k<18$ and $n>\frac{1}{2}e^{2k/e-2}+\frac{1}{2}$.

If $12\le k\le 17$, then $\frac{1}{2}e^{2k/e-2}>2(k+3)^2$. This implies that $\frac{n}{k+3}>2k+6$. We can check by computer that for every integer $n\in(\frac{1}{2}e^{2k/e-2},23000)$, there is a prime number p such that

$$\frac{n}{k+3}<p\le\frac{n}{k} \text{ and } p\nmid(k^2+5k+5)(3k^2+11k+9).$$

Hence by Lemma 2.4, we know that $S_k(n)$ is not an integer for $12\le k\le 17$ and $n<23000$.

If $2\le k\le 11$, then $400>2(k+3)^2$. It can be checked by computer that for every integer $400\le n<23000$, there is a prime number p such that

$$\frac{n}{k+3}<p\le\frac{n}{k} \text{ and } p\nmid(k^2+5k+5)(3k^2+11k+9).$$

Note that for the above prime p, we have $p > \frac{n}{k+3} > 2k+6$. Then Lemma 2.4 tells us that $S_k(n)$ is not an integer if $2 \le k \le 11$ and $n \ge 400$.

For the remaining case $n < 400$, we can verify by using Maple 12 that $S_k(n)$ is not an integer for any integers $2 \le k \le 11$ and $n < 400$.

This completes the proof of Theorem 3.1. $\square$

Reference

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